SHEAF THEORY seminar -

Cohomology of real smooth manifolds

OVERVIEW:

1) De Rham cohomology and the comparison with sheaf cohomology:

Let X be an n-dimensional (real) smooth manifold.

Let Ω^k_X denote the real vector space of (smooth) differential forms of degree k, $k \in \mathbb{N}_0$, on X.

The exterior differential is a morphism $d: \Omega^k_X \to \Omega^{k+1}_X$ $\forall k$.

We note that:

$\operatorname{im}(d: \Omega^{k-1}_X \to \Omega^k_X) = \{\omega$ smooth k-form on $X \mid \omega = d\eta$, i.e. exact, for $\eta \in \Omega^{k-1}_X\}$

$\subset \ker(d: \Omega^k_X \to \Omega^{k+1}_X) = \{\omega$ smooth k-form on $X \mid d\omega = 0$, i.e. closed$\}$.

We also know that: $d^2 = 0$.

Therefore, we define:

$$H^k_{dR}(X, \mathbb{R}) = \frac{\ker(\Omega^k_X \to \Omega^{k+1}_X)}{\operatorname{im}(\Omega^{k-1}_X \to \Omega^k_X)},$$

the kth de Rham cohomology group. (It is a real vector space, but because other cohomology theories produce only groups it is traditional to use the term group in this context as well.)

Since $\Omega^k_X = 0$ for $k<0$ and $k>\dim X$, it is clear that the cohomology is zero in such degrees as well.

For $0 \leq k \leq n$, it holds:

$H^k_{dR}(X) = 0 \iff$ every closed k-form on X is exact.

- The de Rham resolution:

The sheaf we will now be working with, is the constant sheaf $\underline{\mathbb{R}}$ of stalk $\mathbb{R}$, for which we want to construct a particular resolution.

Let A^k, $k \in \mathbb{N}_0$, be the sheaf of smooth k-forms on X. This is a sheaf of sections of a vector bundle over X and thus forms a sheaf of modules over the sheaf of real smooth functions (Voisin I.4, 4.6).

The exterior differential $d: A^k \to A^{k+1}$ is then a morphism of sheaves $\forall k$.

• PROPOSITION 1:

The complex $0 \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \to \dots \to \mathcal{A}^n \to 0$ forms a resolution of $\underline{\mathbb{R}}$.

Proof: This is essentially due to Poincaré's lemma, which says that, for $U \underset{open}{\subset} \mathbb{R}^n$ star-shaped, then every smooth closed k-form, $0 \leq k \leq n$, is exact. This means that, on X, a closed k-form is locally exact (since, for instance, X admits a (differentiable) good open cover). That is, $\mathcal{A}^{k-1} \xrightarrow{d} \mathcal{A}^k \xrightarrow{d} \mathcal{A}^{k+1}$ is stalk-wise exact in the middle for $k \geq 1$ and thus, exact as a sequence of sheaves.

Moreover, $\ker(d: \mathcal{A}^0 = \mathcal{C}^\infty_{X,\mathbb{R}} \to \mathcal{A}^1) =$ {locally constant functions}, so that

$0 \to \underline{\mathbb{R}} \hookrightarrow A^0 \xrightarrow{d} \dots \to A^n \to 0$ is exact.

• PROPOSITION 2:

$\forall k \in \mathbb{N}_0$: $H^k(X, \underline{\mathbb{R}}) = H^k_{dR}(X, \mathbb{R})$.

Proof: We use the de Rham resolution of $\underline{\mathbb{R}}$. Since the sheaf A^0 admits partition of unity, any A^0-Module is a fine sheaf; in particular, A^k is fine $\forall k$. This implies that $H^i(X, A^k) = 0$ $\forall i > 0$ (Voisin, I.4, 4.36), which means that the de Rham resolution is acyclic. Thus (Voisin, I.4, 4.32), $H^k(X, \mathbb{R})$ is equal to the cohomology of the complex of the global sections of the real de Rham complex.

2) Singular cohomology and the comparison with sheaf cohomology:

For a topological space X, singular cohomology (defined over $\mathbb{Z}$) is the cohomology of the complex of the singular cochains $C^k_{sing}(X;\mathbb{Z})$, i.e. the dual of the complex of the singular chains $(C_k(X;\mathbb{Z}),\partial)$, $k\in\mathbb{N}_0$. Let X be locally contractible. We will now proceed just as in the last section.

- The singular resolution:

We now consider the sheaf C^k_{sing} of singular cochains, which is associated to the presheaf $U\mapsto C^k_{sing}(U;\mathbb{Z})$. (This presheaf is not a sheaf.)

The differential δ on each section of singular cochains gives a complex differential $\delta : C^{k}_{sing} \to C^{k+1}_{sing}$.

• PROPOSITION 3:

This complex forms a resolution of $\underline{\mathbb{Z}}$, the constant sheaf of stalk $\mathbb{Z}$.

Proof: The singular cohomology of a contractible space is zero in positive degree, which means that the complex $C^{k}_{sing}(U;\mathbb{Z})$ is exact in positive degree on each contractible open set U; that is, at the level of the stalks (We can construct the direct limit with respect to a basis of contractible neighbourhoods.) Thus, the complex of sheaves C^k_{sing} is exact in positive degree.

Since X is locally (pathwise) connected, we have that if a map is constant on each component of X, then it is locally constant. Thus,

$\ker(\delta : C^0_{sing} \to C^1_{sing}) = \{\varphi$ constant on each connected component of $X\} = \mathbb{Z}$.

• THEOREM 4:

There is a canonical isomorphism $\forall k \in \mathbb{N}_0$: $H^k(X, \mathbb{Z}) \cong H^k_{sing}(X; \mathbb{Z})$.

Proof: We use the singular resolution of $\mathbb{Z}$. We then note that the underlying presheaf is obviously flabby (*)! However, the sheafification functor doesn't preserve flabbiness (for instance, the constant sheaf isn't, in general, flabby; while the constant presheaf always is). To see that, in this case, that does happen, i.e. that the sheaf of singular cochains is flabby $\forall k$, refer to Sella's paper (listed in the bibliography at the end of the

(*): It is not usual to use this notion for presheaves. It is defined in the obvious way and it makes sense, but the actual point is the observation about the sheafification functor.

notes). Thus, C^k_{sing} is acyclic (Voisin, I.4, 4.34), which means that $H^k(X, \mathbb{Z}) = H^k(\Gamma(C^*_{sing}))$.

It remains to show that the complex $\Gamma(C^*_{sing})$ is quasi-isomorphic to the complex C^*_{sing}. This is done via the theorem of small chains, see [1], [3] in the bibliography.

—

It actually holds:

$H^k(X, \underline{R}) \cong H^k_{sing}(X; R)$ for any commutative ring R.

- The de Rham theorem:

Let X be a smooth manifold. What we have just seen (PROP. 3 + THEOREM 4), is the standard sheaf-theoretic proof of the de Rham theorem, which states that:

$H^k_{dR}(X, \mathbb{R}) \cong H^k_{sing}(X; \mathbb{R})$.

This result expresses a deep connection between the topological and smooth properties of a manifold. (For instance, if one has some information about the topology of a smooth manifold X, the de Rham theorem can be used to draw conclusions about solutions to differential equations such as $d\eta = \omega$ on X. Conversely, if one can prove that such solutions do or do not exist, then one can draw conclusions about the topology.)

One can also prove it without sheaf theory, which, of course, has many other powerful applications. However, in this (somewhat simple) smooth case, the geometric proof gives us a neat description of the isomorphism of the theorem.

Over $\mathbb{R}$, $H^K_{sing}(X)$ can be identified with $\mathrm{Hom}_{\mathbb{R}}(H^{sing}_K(X;\mathbb{Z});\mathbb{R})$, thanks to the universal coefficient theorem for cohomology of algebraic topology. Thus, $H^k_{dR}(X, \mathbb{R}) \cong Hom_{\mathbb{R}}(H^{sing}_k(X; \mathbb{Z}), \mathbb{R})$.

Let ω be a closed k-form on X, a a smooth k-simplex in X. We define the integral of ω over a to be: $\int_a \omega = \int_{\Delta_k} a^* \omega$, which makes sense if we consider Δ_k as a domain of integration of $\mathbb{R}^k$. We then define it linearly for smooth k-chains.

The Stoke's theorem for chains (see [2] in the bibliography) allows us to define a natural linear map:

$$H^k_{dR}(X) \to \mathrm{Hom}_{\mathbb{R}}(H^{sing}_k(X;\mathbb{Z}), \mathbb{R})$$

$$[\omega] \mapsto ([c] \mapsto \int_{\tilde{c}} \omega$$

for $\tilde{c}$ (any) smooth k-cycle representing the homology class $[c]$).

One can check that the map is well-defined and that the isomorphism of the theorem is indeed specifically given by integration.

3) Čech cohomology and the comparison with sheaf cohomology:

For a ball $U \underset{open}{\subset} \mathbb{R}^n$: $H^k(U, \underline{\mathbb{R}}) = 0$, $k>0$, by applying Poincaré's lemma to U and the isomorphism with de Rham cohomology.

A (smooth) manifold X admits a cover $\mathcal{U}$ by open sets U_i, which are homeomorphic to balls, as are all their intersections U_I. Moreover, the de Rham cohomology groups are topological invariants.

Thus (Voisin I.4, 4.41), we have that: $H^k(X, \underline{\mathbb{R}}) = \check{H}^k(\mathcal{U}, \underline{\mathbb{R}})$ $\forall k \in \mathbb{N}_0$.

4) An interpretation of the group H^1

Let X be a separable topological space.

Let F^* be the sheaf of the corresponding multiplicative groups of the sheaf of rings $\mathcal{C}^0_{X,\mathbb{C}}$.

One can represent $\alpha \in H^1(X, F^*)$ by a Čech cocycle (Voisin I.4, 4.3.3), in order to prove the following theorem.

• THEOREM 5:

The group $H^1(X, F^*)$ is in bijection with the set of isomorphism classes of free rank 1 modules over $\mathcal{C}^0_{X,\mathbb{C}}$, and thus also with the isomorphism classes of rank 1 vector bundles equipped with continuous structure.

Proof: Voisin, I.4, 4.49.

—

- BIBLIOGRAPHY:

[1]: Cibotaru D., Sheaf cohomology, online notes 2005

[2]: Lee J.M.: Introduction to smooth Manifolds, Springer 2013

[3]: Sella Y.: Comparison of sheaf cohomology and singular cohomology, arXiv:1602.06674v3 2016

[4]: Voisin C.: Hodge Theory and Complex Algebraic Geometry: Volume I, Cambridge University Press 2002